The Binary Regenerative Channel*

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The nature of the errors in a regenerative digital transmission system is such that a memoryless channel is a poor model for predicting the error phenomena. In this paper we present a model which provides a reasonable approximation to observed error phenomena. The memory of the channel is represented by a Markov model. This model is similar to the model developed by E. N. Gilbert, but several important modifications greatly simplify the estimation of parameters, and make the model correspond more closely to the physical phenomena involved.

Bounds for the channel capacity of the binary regenerative channel are obtained. Error separation, block error, and burst statistics are derived.

Error model parameters are derived from available experimental data on the T1 digital transmission line and the switched telephone network. The Markov model is shown to provide a good representation of the observed error phenomena.

I. INTRODUCTION

The Gilbert burst-noise channel introduced the idea of error states.[1] The error states represent different error processes, each of which generates independent errors. Gilbert's model yields a "renewal error process," that is, an error process for which the gaps between successive errors are independent random variables with the same probability distribution. Elliott[2] introduced a generalization which yields what we shall call a "Markov error process," that is, an error process for which the gaps between errors are dependent random variables with probability distributions which depend only on the last gap between errors. More recently, Elliott used a renewal error process, with component error processes which do not generate independent errors.[3] In order to match experimental data for block

error distributions, he found it necessary to introduce three empirically derived error separation distributions.

Berger and Mandelbrot proposed a renewal error process in which the error separation follows a Pareto distribution.[4] Sussman has used the Pareto distribution to model the switched telephone network.[5]

Gilbert's model and Elliott's model both assume that the transition probabilities for the error states are independent of the occurrence of errors. In this paper we drop that assumption to define a general Markov error process. We then consider a particular Markov error process in which transitions between error states are allowed only when an error occurs. We associate this error process with the "binary regenerative channel." Error separation, block error, and burst statistics are derived for this latter process. Error model parameters are calculated from available data for the T1 digital transmission line, and for the switched telephone network. We discuss briefly the possible usefulness of the Pareto distribution for approximating a many-state Markov error process, or for approximating a nonstationary error process.

This author extends this model to apply to a ternary channel.[6]

II. ERROR MODEL

An error model must be able to reproduce the burst error phenomena which are known to occur in digital channels. Real channels seldom appear to be memoryless, and it is common for a large fraction of the errors to be burst errors. To reproduce the burst phenomena, we have chosen to use a Markov model similar to Gilbert's.[1] Our model differs from his in two important aspects. First, we have attempted to make the model correspond more closely to the physical phenomena involved by introducing several error-producing states, each with different error rates. Second, transitions between states are allowed only immediately following an error. This assumption greatly simplifies estimation of the parameters of the model, since the number of digits between adjacent errors is determined by a single error state.

The similarities and differences between these models are most easily understood by examining the transitions between error states. We shall restrict our present discussion to *two-state* error processes. We define:

Σ_n = error state for the nth error digit

Z_n = nth error digit = 1 for error and 0 for no error
p_{ij} = Prob $\{\Sigma_n = j \mid \Sigma_{n-1} = i,\ Z_{n-1} = 0\}$
q_{ij} = Prob $\{\Sigma_n = j \mid \Sigma_{n-1} = i,\ Z_{n-1} = 1\}$
P_i = Prob $\{Z_n = 1 \mid \Sigma_n = i\}$ = average error probability (for the
binary symmetric channel) of state i.

The state diagram of the general Markov model is shown in Fig. 1.
A renewal error process is obtained if q_{ij} is independent of i (so that
the next error state is independent of the state which produces the
error), or if P_1 or P_2 is zero (so that only one state produces errors).

The Gilbert burst-error process[1] assumes that $p_{ij} = q_{ij}$ and $P_2 = 0$.
The assumption that $P_2 = 0$ makes this process a renewal error process.
Elliott's generalization[2] assumes $p_{ij} = q_{ij}$ but $P_2 \neq 0$. This process is
a renewal error process only if $q_{11} = q_{21}$ (and $q_{12} = q_{22}$). Our model, the
"binary regenerative channel," assumes that $p_{ij} = \delta_{ij}$, the Kronecker
delta. This process is a renewal error process only if $q_{11} = q_{21}$ (and
$q_{12} = q_{22}$).

Our assumption that state transitions can occur only after errors
$(p_{ij} = \delta_{ij})$ seems reasonable for two reasons. First we hold the opera-
tional viewpoint that all our information comes from the occurrence
of errors, and we might as well assume that nothing changes between
errors. This also provides a practical technique for estimating transi-
tion probabilities from error separation data. Furthermore, this model
seems to be quite "stable" in that extremely small transition proba-
bilities are not encountered in practice, so that statistical estimates
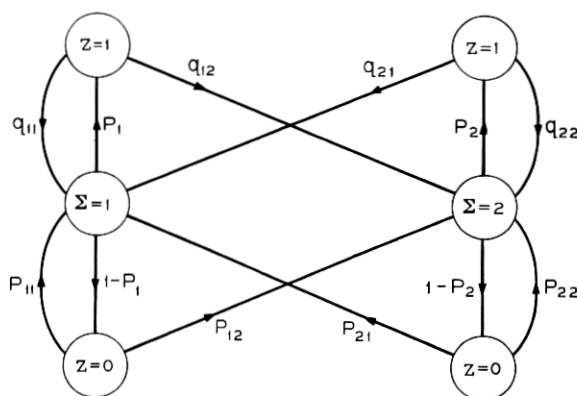are relatively easy to obtain.

Fig. 1 — State diagram of general Markov model.

Second, this is exactly the model we would choose for an error process consisting of random errors plus signal-correlated errors, where a certain fraction of the random errors produce a wake of closely spaced errors. In the case of bursts which are not correlated with random errors, our physical intuition suggests something like $p_{11} = 1$, $p_{12} = 0$ and $p_{21} \approx q_{21} \ll 1$ would be more appropriate. Operationally, however, it really does not matter if the first error in a burst is "incorrectly" identified as a random error.

Although the mathematical descriptions of the Gilbert model, the Elliott model, and our new model are different, we suspect that the error separation, block error, and burst statistics obtained from the three models will be quite similar. (We show that the form of the error separation statistics is identical for the Gilbert model and our new model.) We contend, however, that the new model is more useful because the parameters of the model are easily determined from experimental data and are easier to interpret in terms of physical noise processes.

In the above discussion we have considered the two error states to correspond to different physical error processes in a single channel. However, this single channel is clearly equivalent to a two channel transmission system where the "error" state indicates which channel is being used. We use this latter interpretation in the next section. Notice that the two channels are simply binary symmetric channels with different error rates. In practice we have $P_2 \ll P_1 \approx 1/2$; therefore, we refer to state (channel) 1 as the *burst error state* (channel) and state (channel) 2 as the *random error state* (channel).

III. CHANNEL CAPACITY

Closed form expressions for the capacity of the Markov channel have not yet been found* so that we are limited to determining the capacity for specific numerical values of the parameters. On the other hand, we can find reasonably simple and tight bounds on the capacity which are quite useful. Therefore we consider only bounds on the channel capacity.

Let the sequences of input, output, and error digits be denoted by X_i, Y_i, and Z_i, respectively, with $Y_i \equiv (X_i + Z_i) \bmod 2$ and $i = 1, 2, \cdots$. Since the noise sequence is independent of the input sequence,

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* Note that the method used by Gilbert[1] is valid only for a renewal error process, and did not yield a closed form solution.

the channel capacity is given by

$$C = \max_{p(X)} [H(X) - H(X|Y)]$$

$$= \max_{p(X)} [H(X) - H(Z)] = 1 - H(Z)$$

where H denotes the entropy of the sequence and $p(X)$ is the probability of the sequence X. Following Ash[7] we define the entropy of the noise sequence by

$$H(Z) = \lim_{n \to \infty} H(Z_n \mid Z_{n-1}, \cdots, Z_1).$$

We shall find bounds on the channel capacity by bounding the entropy of the noise, using the steady state probabilities of the Markov noise sequence.

Consider the state diagram in Fig. 1. There are really only four states which we shall designate by $S_i = 11, 10, 21$, or 20, $i = 1, 2, \ldots$, where the first digit indicates which of the two binary symmetric channels is being used ($\Sigma_i = 1$ or 2) and the second digit gives the value of the error digit ($Z_i = 1$ or 0). Thus the state diagram may be redrawn as shown in Fig. 2 (using $p_{ij} = \delta_{ij}$). The steady state probabilities r_{ij} are the solutions of the equations

$$[r_{11} r_{10} r_{21} r_{20}] = [r_{11} r_{10} r_{21} r_{20}] \begin{bmatrix} q_{11}P_1 & q_{11}(1-P_1) & q_{12}P_2 & q_{12}(1-P_2) \\ P_1 & 1-P_1 & 0 & 0 \\ q_{21}P_1 & q_{21}(1-P_1) & q_{22}P_2 & q_{22}(1-P_2) \\ 0 & 0 & P_2 & 1-P_2 \end{bmatrix}.$$

It can be shown that

$$r_{11} = R_1 P_1$$

$$r_{10} = R_1 (1 - P_1)$$

$$r_{21} = R_2 P_2$$

$$r_{20} = R_2 (1 - P_2)$$

where

$$R_1 = P_e \frac{Q_1}{P_1} \qquad Q_1 = \frac{q_{21}}{q_{12} + q_{21}}$$

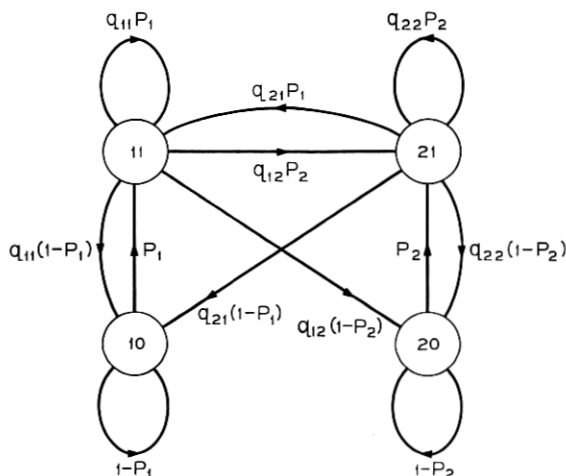$$R_2 = P_e \frac{Q_2}{P_2} \qquad Q_2 = \frac{q_{12}}{q_{12} + q_{21}}$$

Fig. 2 — State diagram of binary regenerative channel.

$$P_e = \frac{1}{\dfrac{Q_1}{P_1} + \dfrac{Q_2}{P_2}} = R_1 P_1 + R_2 P_2 .$$

We notice that R_1 and R_2 are the steady state probabilities of Σ_i, Q_1 and Q_2 are the steady state probabilities of Σ_i given that $Z_{i-1} = 1$, and P_e is the steady state probability that $Z_i = 1$ (that is, P_e is the average error rate).

We are now ready to compute upper and lower bounds on $H(Z)$. An upper bound is

$$H(Z) \leq \lim_{n \to \infty} H(Z_n \mid Z_{n-1})$$

$$= P_e h(Q_1 P_1 + Q_2 P_2) + (1 - P_e) h\left[\frac{P_e}{1 - P_e} (1 - Q_1 P_1 - Q_2 P_2)\right]$$

where $h(P) = -P \log P - (1-P) \log (1-P)$. A simpler (and looser) upper bound is

$$\lim_{n \to \infty} H(Z_n \mid Z_{n-1}) \leq \lim_{n \to \infty} H(Z_n) = h(P_e).$$

Since S_n is determined by a first order Markov process, a lower bound is

$$H(Z) \geq \lim_{n \to \infty} H(Z_n \mid Z_{n-1} , \cdots , Z_1 ; S_{n-1})$$

$$= \lim_{n \to \infty} H(Z_n \mid S_{n-1})$$

$$= R_1 P_1 h(q_{11}P_1 + q_{12}P_2) + R_1(1 - P_1)h(P_1)$$

$$+ R_2 P_2 h(q_{21}P_1 + q_{22}P_2) + R_2(1 - P_2)h(P_2).$$

Using the fact that $h(P)$ is a convex function we obtain a simpler (and looser) lower bound

$$\lim_{n \to \infty} H(Z_n \mid S_{n-1}) \geqq R_1 P_1[q_{11}h(P_1) + q_{12}h(P_2)] + R_1(1 - P_1)h(P_1)$$

$$+ R_2 P_2[q_{21}h(P_1) + q_{22}h(P_2)] + R_2(1 - P_2)h(P_2)$$

$$= R_1 h(P_1) + R_2 h(P_2)$$

$$= \lim_{n \to \infty} H(Z_n \mid \Sigma_n).$$

From the loose bounds we see that the capacity of the binary regenerative channel is greater than the capacity $C = 1 - h(P_e)$ of a binary symmetric channel with the same average error rate, and is less than the capacity $C = R_1[1 - h(P_1)] + R_2[1 - h(P_2)]$ which could be achieved if we always knew which component channel was being used.

A convenient way to describe the channel capacity is to give the probability P_c of the binary symmetric channel with capacity C, that is, $H(Z) = h(P_c)$. From the bounds given above it follows that

$$h^{-1}[R_1 h(P_1) + R_2 h(P_2)] \leqq h^{-1}[\lim_{n \to \infty} H(Z_n \mid S_{n-1})] \leqq P_c$$

$$\leqq h^{-1}[\lim_{n \to \infty} H(Z_n \mid Z_{n-1})] \leqq P_e .$$

For the practical case where $P_2 \ll P_1 \approx 1/2$ and $Q_1 \approx Q_2 \approx 1/2$ the above inequalities are approximately

$$P_2 \lesssim P_2 \lesssim P_c \lesssim (1 - Q_1 P_1)P_e \lesssim P_e$$

or

$$Q_2 P_e \lesssim P_c \lesssim (1 - Q_1 P_1)P_e .$$

The loose bounds given above can be generalized to apply to any finite number of memoryless, nonsymmetric channels in the form

$$B(R_1 P_1 + \cdots + R_m P_m) \leqq C \leqq R_1 B(P_1) + \cdots + R_m B(P_m)$$

where R_i is the steady state probability of using channel i and $B(P_i)$ is the capacity of channel i.

Digital transmission systems may use many digital links, with regeneration of the transmitted signal at the end of each link. For a system of N identical links where $NP_2 \ll P_1$, we can approximate the over-all system by a single digital regenerative link with the same q_{ij} and P_1 but with P_2 replaced by $P_2' = NP_2$. This substitution yields

$$P_e' \approx NP_e, \qquad R_1' \approx NR_1 \qquad R_2' \approx R_2$$

which agrees with one's intuitive notion of how the over-all system should behave. That is, the number of random errors and burst errors both increase by a factor of N, and the length of the bursts remains the same.

IV. ERROR STATISTICS

We shall now derive error separation, block error, and burst statistics for the two-state Markov error process. We assume two component error processes which generate independent errors with different average error rates, P_i, $i = 1, 2$. Transitions between error states are allowed only after errors, with the probabilities given by

$$q_{ij} = \text{Prob \{state } i \rightarrow \text{ state } j | \text{last digit was an error\}}.$$

For the error separation statistics we shall make use of several results for independent errors. We begin by deriving the basic equations for an independent error process. Let P be the probability that any digit is in error. The error sequence then contains a 1 with probability P and O with probability 1-P. Given an error, the probability that the next error occurs on the kth digit is

$$p(k) = \text{Prob } \{O^{k-1}1|1\} = \text{Prob } \{O^{k-1}1\} = P(1 - P)^{k-1}.$$

The average error separation is

$$\bar{k} = \sum_{k=1}^{\infty} kp(k) = P \sum_{k=1}^{\infty} k(1 - P)^{k-1} = \frac{1}{P}.$$

The probability that the number of good digits between errors is greater than or equal to n (that is, the error separation is $n+1$ digits or greater) is given by the cumulative distribution

$$Q(n) = \text{Prob } \{k > n\} = 1 - \sum_{k=1}^{n} p(k)$$

$$= 1 - P \sum_{k=1}^{n} (1 - P)^{k-1} = (1 - P)^n.$$

For $P \ll 1$ the following approximation is quite useful

$$(1 - P)^n = e^{n \, \log(1-P)} \approx e^{-nP}.$$

The probability of getting m errors in n digits is

$$P(m, n) = \frac{n!}{m! \, (n - m)!} P^m (1 - P)^{n-m}.$$

The probability that a block of n digits contains an error is

$$P(\geq 1, n) = 1 - P(0, n) = 1 - (1 - P)^n \approx nP \text{ for } P \ll 1, \ n \ll \frac{1}{P}.$$

Let Q_j, $j = 1, 2$, be the unconditional probability of being in state j at the first digit following an error. Making use of the results for the independent error process, we have

$$p(k) = Q_1 P_1 (1 - P_1)^{k-1} + Q_2 P_2 (1 - P_2)^{k-1}$$

$$\bar{k} = \frac{Q_1}{P_1} + \frac{Q_2}{P_2} \equiv \frac{1}{P_e}$$

$$Q(n) = Q_1 (1 - P_1)^n + Q_2 (1 - P_2)^n$$

where P_e is the average error rate.

The expression for $P(m, n)$ for the Markov error process is a very complicated function of the parameters of the process. However, the form of the dependence upon n is easily found through an appropriate set of recurrence relations. The recurrence relations are also useful for computing numerical values on a digital computer.

Let $A_i(m, n)$ be the probability that m errors have occurred (that is, m occurrences of state 11 or 21) in n digits and that channel i is used for the $n+1$st digit (that is, $\Sigma_{n+1} = i$). Then

$$P(m, n) = A_1(m, n) + A_2(m, n).$$

Considering all possible events which may occur at the nth digit we obtain the following pair of recurrence relations.

$$A_1(m, n) = A_1(m, n - 1) \cdot (1 - P_1) + A_1(m - 1, n - 1) \cdot P_1 q_{11}$$
$$+ A_2(m - 1, n - 1) \cdot P_2 q_{21}$$
$$A_2(m, n) = A_2(m, n - 1) \cdot (1 - P_2) + A_2(m - 1, n - 1) \cdot P_2 q_{22}$$
$$+ A_1(m - 1, n - 1) \cdot P_1 q_{12} .$$

Solving the equations for successive values of m we find that the

general solution has the form[6]

$$P(m, n) = G_m(n)(1 - P_1)^{n-m} + G_m^*(n)(1 - P_2)^{n-m}$$

where $G_m(n)$ is a polynomial of degree m in the variable n, and the asterisk denotes a cyclic permutation of the parameter subscripts, that is, $1 \to 2$ and $2 \to 1$. Assuming that the Markov process is in the steady state at digit "zero," the first two polynomials are

$$G_0(n) = R_1$$

$$G_1(n) = 2\left(\frac{1 - P_1}{P_2 - P_1}\right)R_1P_1q_{12} + [R_1P_1q_{11}]n.$$

It is possible to determine the functional dependence of $P(m, n)$ upon n because we assumed that the error state changes only after an error occurs. This effectively decouples the set of recurrence relations so that $A_1(m, n)$ and $A_2(m, n)$ can be determined separately. For larger m the explicit expressions for the coefficients of $G_m(n)$ become so complicated that they are of little use. Thus one can only hope to gain some insight into the behavior of $P(m, n)$ as a function of m by numerical evaluation for a typical case.

The average number of digit errors in a block of n digits is

$$\bar{m} = \sum_{i=0}^{n} iP(i, n) = nP_e .$$

Given that the block contains one or more errors, the average number of digit errors is

$$\bar{e} = \frac{\bar{m}}{P(\geq 1, n)} = \frac{n}{Q_1[1 - (1 - P_1)^n]/P_1 + Q_2[1 - (1 - P_2)^n]/P_2} .$$

In practice we usually have $P_2 \ll P_1 \approx 1/2$ and $R_1 \ll R_2 \approx 1$. Therefore, we have

$$P(m, n) \approx G_m^*(n)e^{-(n-m)P_2} \quad \text{for} \quad n - m \gg 1/P_1 .$$

Specifically, we find that

$$P(0, n) \approx e^{-nP_2}$$

$$P(\geq 1, n) \approx 1 - e^{-nP_2}$$

$$\bar{e} \approx \frac{1}{Q_2} \cdot \frac{nP_2}{1 - e^{-nP_2}} \quad \text{for} \quad n \gg 1/P_1 .$$

The burst error behavior of the channel is indicated by the num-

ber of successive occurrences of state 1, the burst error state. Let

$$p(b) = \text{Prob \{leave state 1 after } b \text{ errors|now in state 1\}}$$
$$= q_{11}^{b-1}(1 - q_{11}) \qquad b = 1, 2, \cdots .$$

Notice that b occurrences of state 1 implies $b + 1$ "closely spaced" errors.

In the next section we calculate error model parameters from available data on transmission of binary information over the T1 digital transmission line, and over the switched telephone network. A three-state model is required to provide a reasonable match in some cases. Therefore, we digress briefly to generalize our results to apply to a three-state model. (Actually, we give the results in a form which is suitable for any finite number of states.)

We assume three component error processes which generate independent errors with different average error rates, P_i, $i = 1, 2, 3$. Transitions between error states are allowed only after errors, with the probabilities given by

$$q_{ij} = \text{Prob \{state } i \rightarrow \text{ state } j | \text{last digit was an error\}}.$$

Let Q_j, $j = 1, 2, 3$, be the unconditional probability of being in state j at the first digit following an error. The Q_j are the solutions of the following set of equations:

$$Q_1 = Q_1 q_{11} + Q_2 q_{21} + Q_3 q_{31}$$
$$Q_2 = Q_1 q_{12} + Q_2 q_{22} + Q_3 q_{32}$$
$$Q_3 = Q_1 q_{13} + Q_2 q_{23} + Q_3 q_{33} .$$

Corresponding to the previous results we now have

$$p(k) = Q_1 P_1 (1 - P_1)^{k-1} + Q_2 P_2 (1 - P_2)^{k-1} + Q_3 P_3 (1 - P_3)^{k-1}$$
$$\bar{k} = \frac{Q_1}{P_1} + \frac{Q_2}{P_2} + \frac{Q_3}{P_2} \equiv \frac{1}{P_e}$$
$$Q(n) = Q_1 (1 - P_1)^n + Q_2 (1 - P_2)^n + Q_3 (1 - P_3)^n$$

where P_e is the average error rate. The recurrence relations become

$$A_1(m, n) = (1 - P_1) A_1(m, n - 1) + P_1 q_{11} A_1(m - 1, n - 1)$$
$$\qquad\qquad + P_2 q_{21} A_2(m - 1, n - 1) + P_3 q_{31} A_3(m - 1, n - 1)$$
$$A_2(m, n) = (1 - P_2) A_2(m, n - 1) + P_2 q_{22} A_2(m - 1, n - 1)$$

$$+ P_3 q_{32} A_3(m - 1, n - 1) + P_1 q_{12} A_1(m - 1, n - 1)$$

$$A_3(m, n) = (1 - P_3) A_3(m, n - 1) + P_3 q_{33} A_3(m - 1, n - 1)$$

$$+ P_1 q_{13} A_1(m - 1, n - 1) + P_2 q_{23} A_2(m - 1, n - 1)$$

where $A_i(m, n)$ is the probability that m errors have occurred and the error process is in state i after n digits. The solution of the recurrence relations gives

$$P(m, n) = A_1(m, n) + A_2(m, n) + A_3(m, n)$$

$$= G_m(n)(1 - P_1)^{n-m} + G_m^*(n)(1 - P_2)^{n-m} + G_m^{**}(n)(1 - P_3)^{n-m}$$

where $G_m(n)$ is a polynomial of degree m in the variable n. The coefficients of $G_m(n)$ are complicated functions of the model parameters. $G_m^*(n)$ is $G_m(n)$ with the parameter subscripts cyclically permuted, that is, $1 \to 2$, $2 \to 3$, $3 \to 1$. $G_m^{**}(n)$ is $G_n^*(n)$ with the same cyclic permutation. We again have $G_o(n) = R_1$ so that

$$P(0, n) = R_1(1 - P_1)^n + R_2(1 - P_2)^n + R_3(1 - P_3)^n$$

$$P(\geq 1, n)$$

$$= 1 - P(0, n)$$

$$= P_e \left[Q_1 \frac{1 - (1 - P_1)^n}{P_1} + Q_2 \frac{1 - (1 - P_2)^n}{P_2} + Q_3 \frac{1 - (1 - P_3)^n}{P_3} \right].$$

The probability of being in state i at any digit is

$$R_i = \frac{Q_i \bar{k}_i}{Q_1 \bar{k}_1 + Q_2 \bar{k}_2 + Q_3 \bar{k}_3} = \frac{\dfrac{Q_i}{P_i}}{\dfrac{Q_1}{P_1} + \dfrac{Q_2}{P_2} + \dfrac{Q_3}{P_3}} = P_e \frac{Q_i}{P_i}.$$

V. EXPERIMENTAL PARAMETERS

5.1 T1 Digital Transmission Line

For the T1 digital transmission line* (see Refs. 8 and 9), the error data was obtained by measurements[10] on three different lines, each looped to obtain an equivalent system length of about 24 miles. In total, there were five runs of approximately one hour duration, that is, about 5×10^9 digits each. The transmitted pattern was 10000000 repeated. Each run produced about 100 errors. The data were proc-

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* Manufactured for Bell System use only, by Western Electric Co., manufacturing and supply unit of the Bell System.

essed in real time with an IBM 7094 computer equipped with a direct data device. The results (a sequence of numbers $a_1, a_2, \ldots$ where a_i is the number of good digits between the jth and $[j + 1]$st errors) were recorded on a magnetic tape.

To determine the parameters of the Markov error process model, we processed the experimental results as follows.

(i) Add 1 to each a to get k, the error separation.
(ii) Classify each k as state
  1 = "burst error state" $1 \leqq k < 10$
  2 = "intermediate error state" $10 \leqq k < 10^3$
  3 = "random error state" $10^3 \leqq k$.
(iii) For each state i, find the average error separation, $\bar{k}_i = 1/P_i$.
(iv) From the sequence of states find the relative frequency of occurrence of state i, Q_i, and the relative frequency of occurrence of a transition from state i to state j, q_{ij}.

Steps iii and iv were carried out for each run individually, and with all runs together (considered as one big sample). Table I lists the parameters (rounded to two significant digits) which were obtained by the above procedure. Notice that the conditions $q_{11} = q_{21} =$

TABLE I — MARKOV MODEL FOR T1

Run	q_{ij}			Q_i	P_i	P_e
1	.35	.00	.65	.23	.46	1.5×10^{-8}
	.43	.07	.50	.19	3.2×10^{-3}	
	.12	.31	.57	.58	$.86 \times 10^{-8}$	
2	.35	.26	.39	.36	.46	1.4×10^{-8}
	.67	.08	.25	.19	4.0×10^{-3}	
	.24	.17	.59	.45	$.61 \times 10^{-8}$	
3	.29	.33	.38	.35	.37	1.0×10^{-8}
	.62	.19	.19	.23	4.9×10^{-3}	
	.24	.17	.59	.42	$.44 \times 10^{-8}$	
4	.53	.22	.25	.38	.53	2.4×10^{-8}
	.50	.15	.35	.20	3.6×10^{-3}	
	.19	.20	.61	.42	1.0×10^{-8}	
5	.47	.13	.40	.19	.26	1.6×10^{-8}
	.31	.31	.38	.17	3.7×10^{-3}	
	.08	.14	.78	.64	1.0×10^{-8}	
All runs together	.42	.21	.37	.31	.43	1.6×10^{-8}
	.51	.16	.33	.20	3.7×10^{-3}	
	.16	.20	.64	.49	$.77 \times 10^{-8}$	

q_{31} and $q_{12} = q_{22} = q_{32}$ and $q_{13} = q_{23} = q_{33}$ are not satisfied. Hence, the T1 error process does not appear to be a renewal error process. (This does not mean that we should discard the renewal error process. In essence, it is a first approximation to a real error process, the Markov error process is a second approximation, and higher order Markov processes are higher order approximations. The first approximation may be satisfactory in some applications.) Using the parameters of Table I, the validity of the model was checked in three ways.

First, the theoretical cumulative distribution of the error separation,

$$Q(n) = Q_1(1 - P_1)^n + Q_2(1 - P_2)^n + Q_3(1 - P_3)^n$$

was plotted for each run. The theoretical and experimental curves matched within approximately ±0.05, for all five runs. Typical curves are shown in Fig. 3a (semilog plot) and 3b (log-log plot). Notice that we could have derived rough values for the Q_i and P_i by inspection of the experimental $Q(n)$ curve.
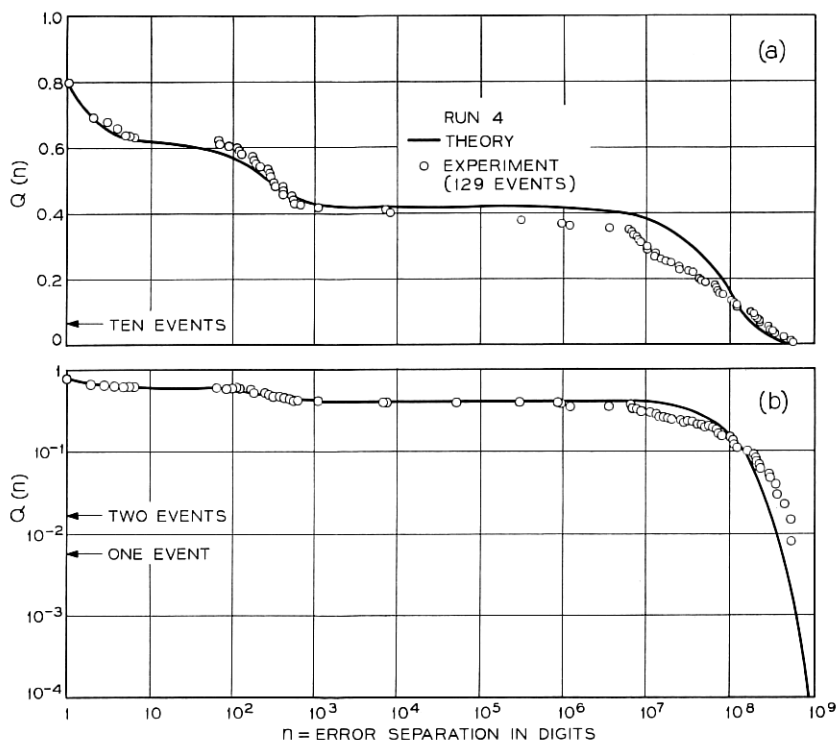

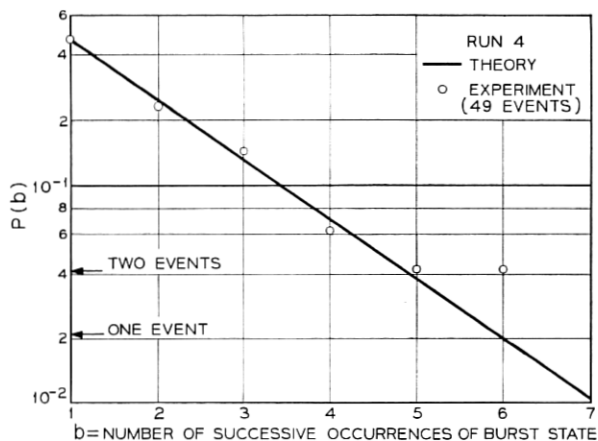Fig. 3 — Error separation statistics for T1 digital transmission line.

Fig. 4 — Burst statistics for T1 digital transmission line.

Second, to check the burst error behavior of the channel, we compared the experimental and theoretical probability densities, $p(b)$, of successive occurrences of the burst error state

$$p(b) = q_{11}^{b-1}(1 - q_{11}).$$

The agreement is excellent as illustrated in Fig. 4. (Since our sample size is only 49, we should not expect the experimental points to follow the theoretical curve for probabilities of about $1/49 \approx 0.02$.) The experimental and theoretical curves matched within approximately 0.02 in all five runs. Notice that the procedure for calculating q_{11} simply provides an exact match at $b = 1$.

Third, to check the adequacy of the model for predicting block error statistics, we compared the theoretical and experimental (averaged over all possible phases) curves for $P(m, n)$. Figures 5 and 6 show $P(m, n)$ versus n and m, respectively. The agreement is excellent for $m \leq 4$. For $m \geq 5$ the experimental curves are somewhat erratic owing to the small sample (the quantum of probability is approximately 2×10^{-10} in this case), which happened to contain two unusual error patterns.

The excellent match between the experimental data and the model indicates that a three-state Markov error process with independent transitions is a good representation of the T1 error process. Since this is the case, it is useful to consider a physical interpretation of the mathematical model. The three different error states correspond
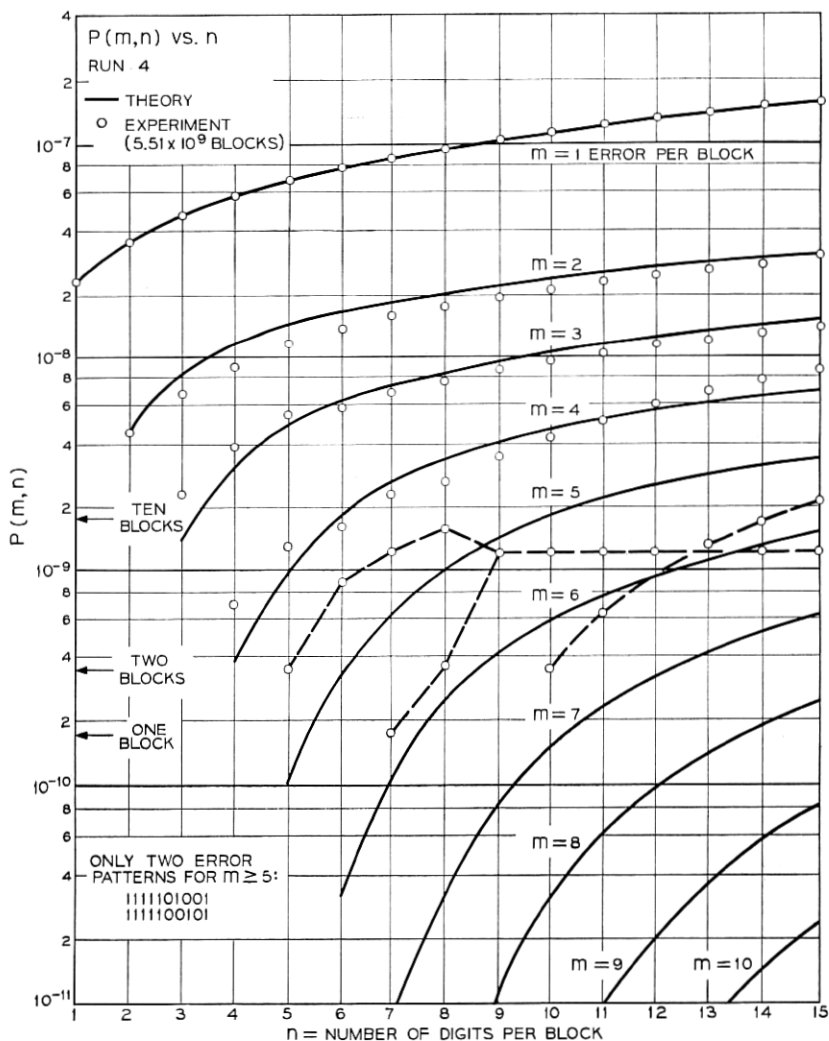
Fig. 5 — Block error statistics (digits per block) for T1 digital transmission line.

to different sources of error, of which only one is controlling at any given time. Allowing state transitions only at error digits corresponds with the fact that we cannot identify the controlling error process except by the error (and error separation) which it produces.

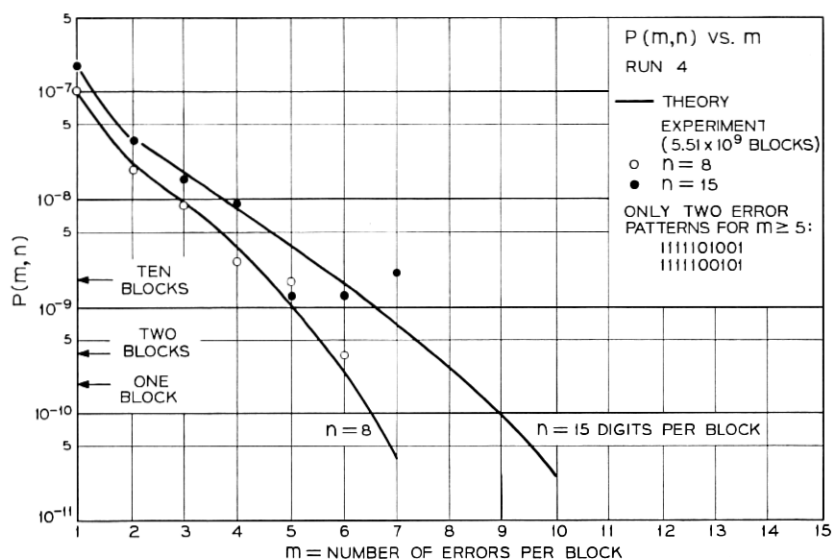As for the sources of error, we can make several speculations. Fur-

Fig. 6 — Block error statistics (errors per block) for T1 digital transmission line.

ther experimental data will be required to determine which of the suggested possibilities is correct. "Burst errors" may result from signal correlated errors (generated in successive regenerators) following a random error, or from burst-like interferences such as impulse noise (all errors of the burst generated at the same regenerator). "Intermediate errors" may be caused by looping effects (outgoing and incoming regenerators are packaged together and are thus subject to the same interference), or by slowly propagating interferences such

TABLE II—SIGNAL DEPENDENCE OF T1 ERRORS

Run	Number of errors for each signal digit								Total
	1	0	0	0	0	0	0	0	
1	0	31	13	6	8	5	4	7	74
2	1	19	8	12	6	7	6	6	65
3	1	26	13	7	5	9	7	2	70
4	2	63	15	11	10	11	8	10	130
5	1	54	7	3	4	4	1	4	78
All runs	5	193	56	39	33	36	26	29	417
Percent all runs	1.2	46.3	13.4	9.4	7.9	8.6	6.2	7.0	100.0

as teletype or other dc signaling. "Random errors" are assumed to come from thermal noise.

We have thus far ignored the question of whether the error sequence is really independent of the transmitted signal sequence. Table II summarizes the available experimental data concerning this point. We observe that the average probability of error is roughly 5 to 40 times greater when the signal digit $X_i = 0$, depending on the number of digits since the last 1. (Notice, however, that Table II does not really tell us the number of digits since the last 1 because errors cause the "signal" to be different in successive regenerators.) Over all, the average probability of error is roughly 10 times greater when $X_i = 0$.

What does this imply about our model? First, since there are so few errors when $X_i = 1$, our component channels are nonsymmetric and the model parameters derived above essentially apply only when $X_i = 0$. In fact, it is possible that the error rate for $X_i = 1$ is the same in all three component channels, so that no burst phenomena occurs if $X_i = 1$ for all i. Second, the dependence of the error rate on the number of digits since the last 1 probably results from intersymbol interference. This suggests that bursts might very well be signal correlated errors which are generated in successive regenerators, in which case the average length of a burst should increase with the number of regenerators. Unfortunately, the available data are not sufficient to verify or disprove these conjectures.

How do we correct our model to take into account the data presented in Table II? As a first approximation we would replace the three component binary symmetric channels with memoryless nonsymmetric binary channels with error probabilities P_1, P_2, and P_3 for $X_i = 0$, and P_1', P_2', and P_3' for $X_i = 1$. With the limited data available the best we can do is to use the previously calculated values for P_1, P_2, and P_3, and let (using the figures for all runs)

$$P_1' = P_2' = P_3' = \tfrac{5}{417}(1.6 \times 10^{-8}) = 1.9 \times 10^{-10}.$$

The computation of channel capacity and error statistics now becomes more difficult because we must consider the joint probability densities of the source and channel. However, we can still use the bounds for channel capacity given at the end of Section III.

To get any better approximation we must replace the three channels with three nonsymmetric binary channels with memory. The memory would contain d, the number of digits since the last 1, and could probably

be limited to three states: $d = 1$, $d = 2$, and $d \geq 3$. (As already discussed this memory would also generate burst phenomena so that we might possibly require only two channels for this model.) This approach is intuitively appealing for modeling the effects of intersymbol interference, but we should use ternary channels with memory because the T1 digital transmission line actually transmits ternary signals.

The fact that the average error probability is always greater for $X_i = 0$ (that is, even for $d \geq 3$) probably is because of long term intersymbol interference, which in the case of T1 may persist for hundreds of digits. This interference is approximately proportional to the running sum of the digits $W_i = X_1 + X_2 + \ldots + X_i$. The Bipolar Code used in the T1 digital transmission system guarantees that W_i can assume only the values 0 or -1 in the absence of errors.* We assume that the output of each regenerator is recoded into Bipolar so that the W_i satisfy the same constraint in successive links, and the channel can be described using a finite memory. Recoding allows one to localize errors to a particular digital link and reduces the error rate in successive links. If the output is not recoded, the W_i are theoretically unbounded which requires an infinite memory to describe the channel.

To summarize our thoughts on the T1 error process, we may say the following. The Markov model analyzed in the preceding sections of this paper provides a good representation of the signal-independent error phenomena, and reproduces all the gross error statistics. The extension of the model suggested in this section shows promise of providing a good representation of the signal-dependent error phenomena, and should reproduce the fine grain error statistics; additional data are required to determine the parameters and validity of the suggested extension. Notice that the signal-dependent memory is realized as a simple Markov process when the source digits are independent random variables.

5.2 Switched Telephone Network

We now consider the error model for the switched telephone network. Gilbert[1] has shown that a *two-state* Markov model provides a good approximation to the cumulative error separation distribution for an *individual* digital channel. Although Gilbert used a different model, his theoretical results for error separation are identical in

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* McCullough[6] treats the general class of ternary restricted sum codes for which the digit sum is bounded ($-a \leq W_i \leq b$) for every code sequence.

form to the results of Section IV, that is,

$$Q(n) = Q_1(1 - P_1)^n + Q_2(1 - P_2)^n.$$

We notice, however, that if we choose Q_i and P_i so that the error separation distributions are identical, the "equivalent" binary regenerative channel will have a higher average error rate. When $P \ll hq$ Gilbert's equation (14) becomes

$$Q(n) \approx \left(1 - \frac{p}{Q - hq}\right)(hq)^n + \left(\frac{p}{Q - hq}\right)(1 - P)^n$$

so that

$$P_1 \approx 1 - hq$$

$$P_2 \approx P$$

$$Q_2 \approx \frac{p}{Q - hq}$$

or

$$h \approx \frac{1 - P_1}{1 - Q_2 P_1}$$

$$P \approx P_2$$

$$p \approx Q_2 P_1$$

and

$$P(1) \approx \frac{1 - Q_2}{1 - Q_2 P_1} \frac{P_2}{Q_2} \approx \frac{1 - Q_2}{1 - Q_2 P_1} P_e .$$

In our notation, the parameters for his examples (see Gilbert's Fig. 3) are

Channel 1146:	$Q_1 = 0$	P_1 arbitrary
	$Q_2 = 1,$	$P_2 = 5.4 \times 10^{-3}$
Channel 1296:	$Q_1 = 0.816$	$P_1 = 0.190$
	$Q_2 = 0.184$	$P_2 = 2.57 \times 10^{-3}.$

For an *average* of many digital channels, a *three-state* Markov model can provide a reasonable numerical fit. Q_i and P_i were determined for samples of the Alexander-Gryb-Nast,[11] Townsend-Watts,[12] and Kelly[13] data on the error performance of the switched telephone network. Table III lists the parameters, which were determined by

TABLE III — MARKOV MODEL FOR SWITCHED TELEPHONE NETWORK

$$Q(n) = Q_1(1 - P_1)^n + Q_2(1 - P_2)^n + Q_3(1 - P_3)^n$$

	Alexander-Gryb-Nast	Townsend-Watts	Kelly
Q_1	0.46	0.58	0.75
Q_2	0.22	0.10	0.10
Q_3	0.32	0.32	0.15
P_1	0.544	0.567	0.56
P_2	10^{-2}	10^{-2}	10^{-3}
P_3	10^{-4}	$\ll 10^{-3}$	5×10^{-6}

trial and error matching of the $Q(n)$ curves. It should be obvious that the Q_i and P_i were quantized rather coarsely. Figure 7 shows that the maximum difference between the experimental and theoretical curves is about ± 0.05.

Although the numerical fit is reasonably good, it is evident that the sharp transition of a single independent-error process is not a good match to the gradual slope of the experimental curves at larger error separations. However, the experimental curves represent an average over many different channels. The parameters of the model will vary from channel to channel, resulting in an over-all error process which contains *many* states. Each state will have a small probability of occurrence (Q_i) and a slightly different average error probability
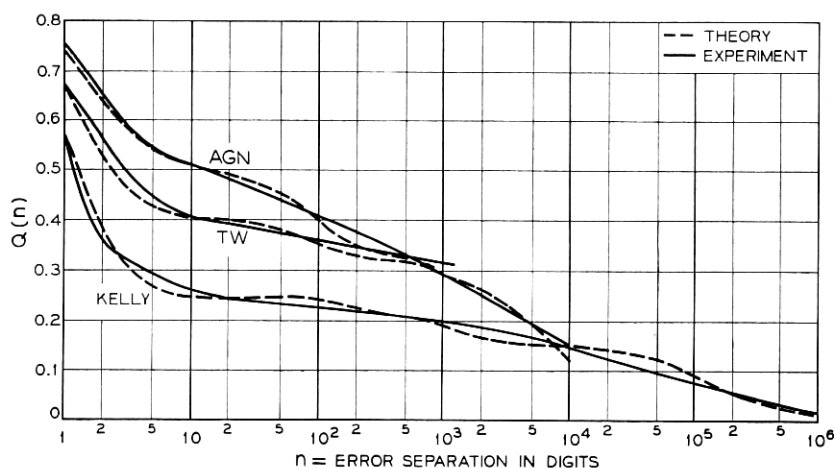

Fig. 7 — Error separation statistics for switched telephone network.

(P_i). $Q(n)$ for such an error process will exhibit a gradual slope at larger error separations.

It is obvious that we could use a larger number of states in the Markov model, and match the experimental $Q(n)$ curves to any desired degree of accuracy. At this point we should consider whether it is an individual channel or an average of many channels that we wish to match. Usually it will be the former, in which case a three-state (or perhaps even a two-state) Markov model will be satisfactory. If it is the latter, it makes sense to seek a single (noninde-pendent) error process which provides a better match for the gradual slope observed at larger error separations. A likely candidate is the Pareto distribution proposed by Berger and Mandelbrot.[4] They also give statistical evidence which supports the renewal error process hypothesis. Sussman[5] has shown that the Pareto distribution pro-vides a good fit to the Alexander-Gryb-Nast data. It is interesting that Sussman hypothesized that the Pareto distribution may be the limiting form of "the superposition of many unrelated error-causing events," which is exactly what our model suggests.

To incorporate the Pareto distribution into our model, we would represent the cumulative distribution of error separation as

$$Q(n) = Q_1(1 - P_1)^n + Q_2(n + 1)^{-\alpha}$$

where α is a parameter which would be chosen so as to give the best match to the experimental data. It should be recognized that the above distribution will not be a good approximation for the Markov error process for very large values of n. As $n \to \infty$ the Markov dis-tribution approaches

$$Q(n) \to Q_\infty(1 - P_\infty)^n$$

where Q_∞ and P_∞ describe the channel with the smallest average random error rate.

In some situations the Pareto distribution may also be a good representation of an individual channel. We have implicitly assumed stationary channels. A nonstationary channel whose parameters vary rapidly with time is essentially equivalent to the average of a large number of stationary channels, each with different parameters. Such a model may be appropriate for digital communication systems using radio links. On the other hand, a slowly varying nonstationary chan-nel is essentially equivalent to a single stationary channel, since the parameters will not change appreciably during any message of rea-

sonable length. This kind of model appears to be appropriate for digital communication systems using paired cable or coaxial cable.

To summarize, we feel that the Markov model is a good representation for the error process of an individual digital channel. The Markov model also explains the observed measurements for the average of a large number of digital channels, and leads naturally to the idea of using the Pareto distribution to approximate the behavior of a Markov error process with many states.

VI. ACKNOWLEDGMENTS

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